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Statistical Properties of a Sum of Sinusoids and Gaussian Noise and its Generalization to Higher Dimensions

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This paper investigates the statistical properties of the sum, $\mathbf{S}$, *of an n-dimensional Gaussian random vector,* $\mathbf{N}$, *plus the sum of M vectors,* $\mathbf{X}_1, \cdots, \mathbf{X}_M$, *having random amplitudes and independent arbitrary orientations in n-dimensional space. We derive expressions for the probability density function (p.d.f.) and distribution function (d.f.) of* $\mathbf{S}$ *and of its length,* $|\mathbf{S}|$, *as series expansions involving only the moments of* $|\mathbf{X}_i|$, $i = 1, \cdots, M$. *In addition, we find the p.d.f. and d.f. of the projection of* $\mathbf{S}$ *onto 1-dimensional space. Our results are generalizations of the* $n = 2$*-dimensional problem of finding the statistical properties of a sum of constant-amplitude sinusoids having independent uniformly distributed phase angles plus Gaussian noise. The latter problem has been treated by Rice[1] and Esposito and Wilson,[2] but our results can also deal with sinusoids having* random *amplitudes. When* $n = 3$, *our findings treat, in the presence of a Gaussian vector, the classical problem of "random flights" dating back to Rayleigh. Some calculations for the 2- and 3-dimensional problem are presented, and an application to coherent phase-shift-keying communications systems is discussed.*

I. INTRODUCTION

In a number of problems arising in communications systems, in multipath phenomena, and in other areas, the determination of the

statistical properties of a sum of sinusoids and Gaussian noise is important for evaluating system performance. For this reason there has been interest in this problem for a number of years. Rice[1] first investigated the statistical properties of the sum of a single constant-amplitude sinusoid and Gaussian noise. Later, Esposito and Wilson[2] considered this same problem but with two constant-amplitude sinusoids having independent uniformly distributed phase angles. More recently, Rice[3] studied the properties of a sum of M sinusoids and Gaussian noise. In this paper, we look at the natural generalization of this problem to n-dimensional space; namely, we determine the statistical properties of the sum of an n-dimensional Gaussian random vector plus the sum of M vectors having random amplitudes and independent arbitrary orientations in n-dimensional space. In the special case when $n = 2$, our results are applicable to the type of problems considered by Rice and Esposito and Wilson, but they can also deal with any number of sinusoids with *random* amplitudes. When $n = 3$, our findings treat, in the presence of a Gaussian vector, the classical problem of "random flights" dating back to Rayleigh.

In Section II we give a definition of spherically symmetric random n-vectors and state a theorem which characterizes the form of such vectors in an n-dimensional spherical coordinate system. We consider M independent spherically symmetric vectors, $\mathbf{X}_1, \cdots, \mathbf{X}_M$, and define $\mathbf{S} = \sum_{i=1}^{M} \mathbf{X}_i$. Using our characterization theorem, we show that the even moments, $E[|\mathbf{S}|^{2k}]$, $k = 1, 2, \cdots$, can be easily expressed in terms of only the moments of $|\mathbf{X}_i|$, $i = 1, \cdots, M$. Then with the normal vector $\mathbf{N} \sim \mathfrak{N}(\mathbf{0}, \sigma^2\mathbf{I})$ independent of the $\mathbf{X}_i$'s, we derive in Section III the probability density functions (p.d.f.'s) and distribution functions (d.f.'s) of $\mathbf{S} + \mathbf{N}$ and of $|\mathbf{S} + \mathbf{N}|$ as series expansions involving the moments of $|\mathbf{S}|$. In addition, we derive the p.d.f. and d.f. of the projection of $\mathbf{S} + \mathbf{N}$ onto 1-dimensional space in terms of a similar series expansion. When $n = 2$ and $M = 2$, we check that our results agree with those of Esposito and Wilson for two constant-amplitude sinusoids.

Last, in Section IV we present some calculations for the 2- and 3-dimensional problems, and discuss some aspects of the computational procedure that we use. Certain of these calculations provide results for the probability of error of a binary coherent phase-shift-keying communications system operating in the presence of M co-channel interferers and Gaussian noise. These results extend previously published computations.[4,5] Additionally, our findings can be used to find

the probability of error of this system operating in the presence of more general types of interference.

II. SPHERICAL SYMMETRY

The generalization of sinusoids with uniformly distributed phase angles are "spherically symmetric" vectors defined in the following way (see Refs. 6 and 7):

Definition: *A* random n-vector $\mathbf{X} = (X_1, \cdots, X_n)$, $n \geqq 1$, is *spherically symmetric with matrix* $\boldsymbol{\varrho}$ if and only if the covariance matrix of $\mathbf{X}$ exists,* $E(\mathbf{X}) = \mathbf{0}$, and the joint characteristic function of $\mathbf{X}$ is of the form:†

$$\Phi_{\mathbf{X}}(\mathbf{u}) = E[e^{j\mathbf{u}\mathbf{X}'}] = h[(\mathbf{u}\boldsymbol{\varrho}\mathbf{u}')^{\frac{1}{2}}] \tag{1}$$

for some function h on $[0, \infty)$ and where $\boldsymbol{\varrho}$ is some $n \times n$ (symmetric) positive definite matrix.‡ Actually, h and $\boldsymbol{\varrho}$ are defined only up to positive multiplicative factors. However, in this paper we are only concerned with spherically symmetric vectors with $\boldsymbol{\varrho} = \mathbf{I} =$ identity matrix. Then h is uniquely determined and $\Phi_{\mathbf{X}}(\mathbf{u}) = h(|\mathbf{u}|)$. We denote such a spherically symmetric vector by the notation "$\mathbf{X}$ is s.s."

Note that if $\mathbf{X}_1$ and $\mathbf{X}_2$ are two independent s.s. vectors, then clearly $\mathbf{X}_1 + \mathbf{X}_2$ is also s.s.

The probability density function of an s.s. vector $\mathbf{X}$ can be found by Bochner's theorem.[8] If $h(|\mathbf{u}|)$ is absolutely integrable, then the p.d.f. of $\mathbf{X}$ is:

$$p_{\mathbf{X}}(\mathbf{x}) = g_n(|\mathbf{x}|), \tag{2}$$

where

$$g_n(r) = \frac{1}{(2\pi)^{n/2}} \frac{1}{r^{(n-2)/2}} \int_0^\infty h(\lambda)\lambda^{n/2} J_{(n-2)/2}(\lambda r) d\lambda \quad r > 0, \quad n \geqq 1.$$

Thus, if $\mathbf{X}$ is s.s., its p.d.f. is constant over every n-dimensional sphere centered about the origin. This vector is precisely what is meant by a "random flight" in a higher dimensional space.

For our purposes, a more suitable characterization of an s.s. vector is given by the following theorem proved in Ref. 9.

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* Expected value will be denoted by $E(\cdot)$.

† We denote vectors by boldface characters: $\mathbf{u} = (u_1, \cdots, u_n)$. The character $\mathbf{u}'$ is the transpose of $\mathbf{u}$. The norm of $\mathbf{u}$ is denoted $|\mathbf{u}| = (\mathbf{u}\mathbf{u}')^{\frac{1}{2}}$.

‡ For $n = 1$, a spherically symmetric random variable has an even characteristic function, $\Phi_{\mathbf{X}}(u) = h[\rho^{\frac{1}{2}}|u|]$.

Theorem 1: Suppose $\mathbf{X} = (X_1, \cdots, X_n)$, $n \geqq 2$, *is s.s. Then there exists a unique set of random variables* $R \geqq 0$, $\Phi_k \in [0, \pi]$, $k = 1, \cdots, n-2$, $\theta \in [0, 2\pi]$ *for which**

$$
\begin{aligned}
X_j &= R\left(\prod_{k=1}^{j-1} \sin \Phi_k\right) \cos \Phi_j \qquad 1 \leqq j \leqq n-2 \\
X_{n-1} &= R\left(\prod_{k=1}^{n-2} \sin \Phi_k\right) \cos \theta \qquad\qquad\qquad\qquad (3) \\
X_n &= R\left(\prod_{k=1}^{n-2} \sin \Phi_k\right) \sin \theta,
\end{aligned}
$$

and furthermore $(R, \Phi_1, \cdots, \Phi_{n-2}, \theta)$ *are independent and have respective p.d.f.'s:*

$$
\begin{aligned}
p_R(r) &= 2\pi^{n/2}\left[\Gamma\left(\frac{n}{2}\right)\right]^{-1} r^{n-1} g_n(r) \qquad r \geqq 0 \\
p_{\Phi_k}(\phi_k) &= \Gamma\left(\frac{n-k+1}{2}\right)\pi^{-\frac{1}{2}}\left[\Gamma\left(\frac{n-k}{2}\right)\right]^{-1} \sin^{n-1-k}\phi_k \\
&\qquad\qquad\qquad\qquad\qquad 0 \leqq \phi_k \leqq \pi \qquad (4) \\
&\qquad k = 1, \cdots, n-2 \\
p_\theta(\theta) &= \frac{1}{2\pi} \qquad 0 \leqq \theta < 2\pi
\end{aligned}
$$

for the $g_n(\cdot)$ *of* (2).

Conversely, if $(R, \Phi_1, \cdots, \Phi_{n-2}, \theta)$ *are independent and have the p.d.f.'s given by* (4), *and* $\mathbf{X}$ *is defined as in* (3), *then* $\mathbf{X}$ *is s.s.*

The utility of this theorem lies in the fact that the random variables $(R, \Phi_1, \cdots, \Phi_{n-2}, \theta)$ are independent with specified p.d.f.'s. As an immediate corollary, we see from (2), (3), and (4) that:

Corollary 1: Suppose $\mathbf{X} = (X_1, \cdots, X_n)$, $n \geqq 1$, *is s.s. Then its p.d.f. is given by:*

$$p_{\mathbf{X}}(\mathbf{x}) = (2\pi^{n/2})^{-1}\Gamma(n/2)\,\frac{p_{|\mathbf{X}|}(|\mathbf{x}|)}{(|\mathbf{x}|)^{n-1}}.$$

Moreover, for $j = 1, \cdots, n$ *and for all* i,

$$\frac{\Gamma(n/2)E[|\mathbf{X}|^{2i}]}{\Gamma[(n/2)+i]} = \frac{\Gamma(\frac{1}{2})E[|X_j|^{2i}]}{\Gamma(\frac{1}{2}+i)}.$$

Using Theorem 1 we can prove:

Theorem 2: Suppose $\mathbf{X}_1, \cdots, \mathbf{X}_M$ *are independent s.s. n-vectors,* $n \geqq 1$. *Let* $\mathbf{S}_j = \sum_{i=1}^{j} \mathbf{X}_i$, $j = 1, \cdots, M$, *and define*

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* We define $\prod_{k=1}^{0} a_k = 1$.

$$\mu_j^{(2k)} = E[|\mathbf{S}_j|^{2k}], \qquad k = 0, 1, \cdots, \qquad j = 1, \cdots, M$$
$$\nu_\ell^{(2m)} = E[|\mathbf{X}_\ell|^{2m}], \qquad m = 0, 1, \cdots, \qquad \ell = 1, \cdots, M.$$

Put

$$c_{n,2m} \triangleq \frac{B[(2m+1)/2, (n-1)/2]}{B[\frac{1}{2}, (n-1)/2]},$$

where $B(\cdot,\cdot)$ is the beta function. Denote

$$\boldsymbol{\mu}_j^{(2m)} = (\mu_j^{(0)}, \mu_j^{(2)}, \cdots, \mu_j^{(2m)})$$
$$\boldsymbol{\nu}_j^{(2m)} = (\nu_j^{(0)}, \nu_j^{(2)}, \cdots, \nu_j^{(2m)})$$

and define $D_{n,j}$ to be an $(m+1) \times (m+1)$ matrix whose (k, ℓ)th element equals

$$\binom{2\ell-2}{2k-2} \frac{c_{n,2k-2}\, c_{n,2\ell-2k}}{c_{n,2\ell-2}} \nu_j^{(2\ell-2k)}, \qquad \text{if} \quad \ell \geqq k$$

and is 0 if $\ell < k$.

Then, for $j = 2, \cdots, M$, and $m = 0, 1, \cdots,$

$$\mu_j^{(2m)} = \sum_{i=0}^{m} \binom{2m}{2i} \frac{c_{n,2i}\, c_{n,2m-2i}}{c_{n,2m}} \mu_{j-1}^{(2i)} \nu_j^{(2m-2i)}. \tag{5}$$

In matrix form this is

$$\boldsymbol{\mu}_j^{(2m)} = \boldsymbol{\mu}_{j-1}^{(2m)} D_{n,j},$$

so that

$$\boldsymbol{\mu}_j^{(2m)} = \boldsymbol{\nu}_1^{(2m)} D_{n,2} \cdots D_{n,j-1} D_{n,j} \tag{6}$$

for $j = 2, \cdots, M$.

Proof: By Theorem 1 we have for each $\mathbf{X}_i$ a corresponding vector in spherical coordinate space:

$$\mathbf{X}_i \leftrightarrow (R_i, \Phi_{1,i}, \cdots, \Phi_{n-2,i}, \theta_i).$$

Since $\mathbf{S}_j$ is a sum of independent s.s. vectors, it is also s.s., so there are vectors corresponding to it:

$$\mathbf{S}_j \leftrightarrow (P_j, \xi_{1,j}, \cdots, \xi_{n-2,j}, \Psi_j).$$

Note that $\mu_j^{(2k)} = E[P_j^{2k}]$ and $\nu_\ell^{(2m)} = E[R_\ell^{2m}]$ and that

$$\begin{aligned} E[\cos^i \Phi_{1,j}] &= E[\cos^i \xi_{1,j}] \\ &= \frac{\Gamma(n/2)}{\Gamma(\frac{1}{2})\Gamma[\frac{1}{2}(n-1)]} \int_0^\pi \cos^i \alpha \sin^{n-2} \alpha\, d\alpha \\ &= \begin{cases} c_{n,2i} & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd.} \end{cases} \end{aligned} \tag{7}$$

Denote the components of $\mathbf{S}_j$ and $\mathbf{X}_j$ as follows:

$$\begin{aligned}\mathbf{S}_j &= (S_{1,j}, \cdots, S_{n,j})\\ \mathbf{X}_j &= (X_{1,j}, \cdots, X_{n,j}),\end{aligned}$$

so that

$$\begin{aligned}S_{1,j} &= P_j \cos \xi_{1,j}\\ X_{1,j} &= R_j \cos \Phi_{1,j}.\end{aligned}$$

Also,

$$\begin{aligned}E[S_{1,j}^i] &= E[P_j^i \cos^i \xi_{1,j}]\\ &= E[P_j^i]E[\cos^i \xi_{1,j}] = 0\end{aligned}\tag{8}$$

if i is odd, and

$$\begin{aligned}E[S_{1,j}^{2m}] &= E[P_j^{2m}]E[\cos^{2m} \xi_{1,j}]\\ &= \mu_j^{(2m)} c_{n,2m}\end{aligned}\tag{9}$$

and

$$E[X_{1,j}^{2k}] = E[R_j^{2k}]E[\cos^{2k}\Phi_{1,j}] = \nu_j^{(2k)} c_{n,2k}.\tag{10}$$

Noting that $S_{1,j-1}$ is independent of $X_{1,j}$ since $\mathbf{S}_{j-1}$ depends only on $\mathbf{X}_1, \cdots, \mathbf{X}_{j-1}$ which are independent of $\mathbf{X}_j$, the following equalities follow from (8) to (10):

$$\begin{aligned}\mu_j^{(2m)} c_{n,2m} &= E[S_{1,j}^{2m}]\\ &= E\{[S_{1,j-1} + X_{1,j}]^{2m}\}\\ &= \sum_{i=0}^{2m} \binom{2m}{i} E\{[S_{1,j-1}]^i [X_{1,j}]^{2m-i}\}\\ &= \sum_{i=0}^{2m} \binom{2m}{i} E[S_{1,j-1}^i] E[X_{1,j}^{2m-i}]\\ &= \sum_{i=0}^{m} \binom{2m}{2i} E[S_{1,j-1}^{2i}] E[X_{1,j}^{2m-2i}]\\ &= \sum_{i=0}^{m} \binom{2m}{2i} \mu_{j-1}^{(2i)} c_{n,2i}\, \nu_j^{(2m-2i)} c_{n,2m-2i}.\end{aligned}$$

Hence,

$$\mu_j^{(2m)} = \sum_{i=0}^{m} \binom{2m}{2i} \frac{c_{n,2i}\, c_{n,2m-2i}}{c_{n,2m}} \mu_{j-1}^{(2i)} \nu_j^{(2m-2i)}.$$

The vector equation

$$\boldsymbol{\mu}_j^{(2m)} = \boldsymbol{\mu}_{j-1}^{(2m)} D_{n,j} \qquad j = 2, \cdots, M \tag{11}$$

follows immediately from this expression. Since $\boldsymbol{\mu}_1^{(2m)} = \boldsymbol{\nu}_1^{(2m)}$, eq. (11) implies that $\boldsymbol{\mu}_j^{(2m)} = \boldsymbol{\nu}_1^{(2m)} D_{n,2} \cdots D_{n,j-1} D_{n,j}$ for $j = 2, \cdots, M$. For $n \geqq 1$, we prove (5) directly.

In Reference 9 we proved the result:

Theorem 3: If $\mathbf{X}$ *is s.s. and independent of* $\mathbf{N} \sim \mathfrak{N}(\mathbf{0}, \sigma^2\mathbf{I})$, *then the p.d.f. of* $\mathbf{X} + \mathbf{N}$ *is:*

$$p_{\mathbf{X}+\mathbf{N}}(\mathbf{z}) = \frac{1}{2\pi^{n/2}\sigma^2}\Gamma\left(\frac{n}{2}\right)\int_0^\infty dv\, p_{|\mathbf{X}|}(v)\frac{1}{[v|\mathbf{z}|]^{(n-2)/2}} \times \exp\left[-\frac{1}{2\sigma^2}(v^2 + |\mathbf{z}|^2)\right] I_{(n-2)/2}\left(\frac{v|\mathbf{z}|}{\sigma^2}\right), n \geqq 1, \quad (12)$$

where $I_\nu(\cdot)$ *is the* ν*th order modified Bessel function of the first kind.*

From this theorem we obtain the following:

Corollary 2: Suppose $\mathbf{X}$ *is s.s, independent of* $\mathbf{N} \sim \mathfrak{N}(\mathbf{0}, \sigma^2\mathbf{I})$, *and* $\sum_{i=0}^\infty [(1/2\sigma^2)^i/i!]E[|\mathbf{X}|^{2i}] < \infty$ *(for example, if* $|\mathbf{X}|$ *is a bounded random variable). Let* Z *be the projection of* $\mathbf{X} + \mathbf{N}$ *onto 1-dimensional space, i.e.,* Z *is (say) the first component of the vector* $\mathbf{X} + \mathbf{N}$. *Then for* $n \geqq 1$ *the p.d.f.'s of* $\mathbf{X} + \mathbf{N}$, *of* $|\mathbf{X} + \mathbf{N}|$, *and of* Z *are given, respectively, by:*

$$p_{\mathbf{X}+\mathbf{N}}(\mathbf{z}) = \frac{\Gamma(n/2)}{(2\pi\sigma^2)^{n/2}}\exp\left(-\frac{1}{2\sigma^2}|\mathbf{z}|^2\right) \times \sum_{i=0}^\infty \frac{L_i^{[(n-2)/2]}(|\mathbf{z}|^2/2\sigma^2)(-1/2\sigma^2)^i E[|\mathbf{X}|^{2i}]}{\Gamma[(n/2)+i]}, \quad (13)$$

$$p_{|\mathbf{X}+\mathbf{N}|}(v) = \frac{2}{(2\sigma^2)^{n/2}}v^{n-1}\exp\left(-\frac{1}{2\sigma^2}v^2\right) \times \sum_{i=0}^\infty \frac{L_i^{[(n-2)/2]}(v^2/2\sigma^2)(-1/2\sigma^2)^i E[|\mathbf{X}|^{2i}]}{\Gamma[(n/2)+i]}, \quad (14)$$

and

$$p_Z(z) = \frac{\Gamma(n/2)}{(2\pi\sigma^2)^{\frac{1}{2}}}\exp\left(-\frac{z^2}{2\sigma^2}\right) \times \sum_{i=0}^\infty \frac{L_i^{(-\frac{1}{2})}(z^2/2\sigma^2)(-1/2\sigma^2)^i E[|\mathbf{X}|^{2i}]}{\Gamma[(n/2)+i]}, \quad (15)$$

where $L_i^{(\alpha)}(\cdot)$ *are the generalized Laguerre polynomials. In addition, for* $n \geqq 1$ *the "distribution functions" of* $|\mathbf{X} + \mathbf{N}|$ *and of* Z *are given, respectively, by:*

$$\Pr\{|\mathbf{X}+\mathbf{N}| > a\} = \frac{1}{\Gamma(n/2)}\Gamma\left(\frac{n}{2}, \frac{a^2}{2\sigma^2}\right) - \left(\frac{a^2}{2\sigma^2}\right)^{n/2}\exp\left(-\frac{a^2}{2\sigma^2}\right) \times \sum_{i=1}^\infty \frac{L_{i-1}^{(n/2)}(a^2/2\sigma^2)(-1/2\sigma^2)^i E[|\mathbf{X}|^{2i}]}{i\Gamma[(n/2)+i]} \quad (16)$$

and

$$\Pr\{Z > a\} = \tfrac{1}{2}\operatorname{erfc}\left(\frac{a}{(2\sigma^2)^{\frac{1}{2}}}\right) - \frac{a}{2}\frac{\Gamma(n/2)}{(2\pi\sigma^2)^{\frac{1}{2}}}\exp\left(-\frac{a^2}{2\sigma^2}\right)$$
$$\times \sum_{i=1}^{\infty} \frac{L_{i-1}^{(\frac{1}{2})}(a^2/2\sigma^2)(-1/2\sigma^2)^i E[|\mathbf{X}|^{2i}]}{i\Gamma[(n/2)+i]}, \quad (17)$$

where $\Gamma(\cdot,\cdot)$ *is the incomplete gamma function (Ref. 10, p. 337) and erfc* $(\cdot)$ *is the complementary error function.*

Proof: From Ref. 10, p. 242, we have the generating function

$$e^{-t^2}\frac{1}{(tx)^{\alpha}} I_{\alpha}(2tx) = \sum_{i=0}^{\infty} \frac{L_i^{(\alpha)}(x^2)(-1)^i t^{2i}}{\Gamma(\alpha+i+1)}. \quad (18)$$

With $t = v/\sqrt{2\sigma^2}$, $x = |\mathbf{z}|/\sqrt{2\sigma^2}$, and $\alpha = (n-2)/2$, we substitute (18) into (12) to get:

$$p_{\mathbf{X}+\mathbf{N}}(\mathbf{z}) = \frac{1}{2\pi^{n/2}\sigma^2}\Gamma\left(\frac{n}{2}\right)\exp\left[-\frac{1}{2\sigma^2}|\mathbf{z}|^2\right]\int_0^{\infty} dv p_{|\mathbf{X}|}(v)$$
$$\times \frac{1}{(2\sigma^2)^{(n-2)/2}}\sum_{i=0}^{\infty}\frac{L_i^{[(n-2)/2]}(|\mathbf{z}|^2/2\sigma^2)(-1/2\sigma^2)^i v^{2i}}{\Gamma[(n/2)+i]}$$
$$= \frac{\Gamma(n/2)}{(2\pi\sigma^2)^{n/2}}\exp\left(-\frac{1}{2\sigma^2}|\mathbf{z}|^2\right)$$
$$\times \sum_{i=0}^{\infty}\frac{L_i^{[(n-2)/2]}(|\mathbf{z}|^2/2\sigma^2)(-1/2\sigma^2)^i E[|\mathbf{X}|^{2i}]}{\Gamma[(n/2)+i]}, \quad (19)$$

assuming that the interchange of integration and expectation is valid. The second assertion of the corollary follows from Corollary 1 since

$$p_{|\mathbf{X}+\mathbf{N}|}(|\mathbf{z}|) = \frac{2\pi^{n/2}}{\Gamma(n/2)}|\mathbf{z}|^{n-1}p_{\mathbf{X}+\mathbf{N}}(\mathbf{z}). \quad (20)$$

To prove (15) we note that $Z = X_1 + N_1$ is a 1-dimensional s.s. random vector and we apply eq. (13) (with $n = 1$) and the second part of Corollary 1 to obtain the desired result.

Next, to show (16) we integrate (14) over the interval (a, ∞) and utilize the relationships:

$$\int_a^{\infty} v^{n-1}\exp\left(-\frac{v^2}{2\sigma^2}\right)dv = \frac{1}{2}(2\sigma^2)^{n/2}\Gamma\left(\frac{n}{2},\frac{a^2}{2\sigma^2}\right) \quad (21)$$

and for $i \geqq 1$,

$$\int_a^\infty v^{n-1} \exp\left(-\frac{v^2}{2\sigma^2}\right) L_i^{[(n-2)/2]}\left(\frac{v^2}{2\sigma^2}\right) dv$$

$$= -\frac{1}{2} a^n \exp\left(-\frac{a^2}{2\sigma^2}\right) \frac{L_{i-1}^{(n/2)}(a^2/2\sigma^2)}{i}. \qquad (22)$$

Equation (21) is given in Ref. 10, p. 337, and (22) is proved in the appendix.

Finally, to obtain (17) we integrate (15) and use eq. (22) with $n = 1$ and the definition:

$$\operatorname{erfc}(x) = \frac{2}{\pi^{\frac{1}{2}}} \int_x^\infty \exp(-t^2) dt.$$

It remains to justify the various interchanges of integration and expectation or summation. For example, to validate the interchange in (19) it suffices (Ref. 11, pp. 28–29) to show that

$$\sum_{i=0}^\infty \frac{(1/2\sigma^2)^i}{\Gamma[(n/2)+i]} E[|\mathbf{X}|^{2i}] \left| L_i^{[(n-2)/2]}\left(\frac{|\mathbf{z}|^2}{2\sigma^2}\right)\right| < \infty. \qquad (23)$$

Since (Ref. 12, p. 207) $|L_i^{(\alpha)}(y)| \leqq e^{y/2}\Gamma(\alpha + i + 1)/i!\Gamma(\alpha + 1)$, the expression in (23) is less than or equal to:

$$\sum_{i=0}^\infty \frac{(1/2\sigma^2)^i}{\Gamma[(n/2)+i]} E[|\mathbf{X}|^{2i}] \exp\left(\frac{|\mathbf{z}|^2}{4\sigma^2}\right) \frac{1}{i!} \frac{\Gamma[(n/2)+i]}{\Gamma(n/2)}$$

which is finite by hypothesis.

The utility of this corollary lies in the fact that we can evaluate the various p.d.f.'s and d.f.'s knowing only the moments of $|\mathbf{X}|$ and not the entire distribution of $\mathbf{X}$.

III. STATISTICAL PROPERTIES OF THE SUM OF INDEPENDENT SPHERICALLY SYMMETRIC VECTORS AND GAUSSIAN NOISE

For simplicity, we combine the results of Corollary 2 and Theorem 2 into:

Theorem 4: Suppose $\mathbf{X}_1, \cdots, \mathbf{X}_M$ *are independent s.s. n-vectors,* $n \geqq 1$, *with moments* $\nu_\ell^{(2m)} = E[|\mathbf{X}_\ell|^{2m}]$, $m = 0, 1, \cdots$, *and* $\ell = 1, \cdots, M$, *which are also independent of* $\mathbf{N} \sim \mathfrak{N}(\mathbf{0}, \sigma^2\mathbf{I})$. *Let* $\mathbf{S} = \sum_{i=1}^M \mathbf{X}_i$ *and assume that* $\sum_{i=0}^\infty (1/2\sigma^2)^i/E[|\mathbf{S}|^{2i}]i! < \infty$ *(for example, if the* $|\mathbf{X}_i|$*'s are bounded random variables). Let Z be the projection of* $\mathbf{S} + \mathbf{N}$ *onto 1-dimensional*

space. Then the following relations for p.d.f.'s and d.f.'s are valid:

$$p_{\mathbf{S}+\mathbf{N}}(\mathbf{z}) = \frac{\Gamma(n/2)}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2}|\mathbf{z}|^2\right) \times \sum_{i=0}^{\infty} \frac{L_i^{[(n-2)/2]}(|\mathbf{z}|^2/2\sigma^2)(-1/2\sigma^2)^i \mu_M^{(2i)}}{\Gamma[(n/2)+i]}, \quad (24)$$

$$p_{|\mathbf{S}+\mathbf{N}|}(v) = \frac{2}{(2\sigma^2)^{n/2}} v^{n-1} \exp\left(-\frac{1}{2\sigma^2}v^2\right) \times \sum_{i=0}^{\infty} \frac{L_i^{[(n-2)/2]}(v^2/2\sigma^2)(-1/2\sigma^2)^i \mu_M^{(2i)}}{\Gamma[(n/2)+i]}, \quad (25)$$

$$p_Z(z) = \frac{\Gamma(n/2)}{(2\pi\sigma^2)^{\frac{1}{2}}} \exp\left(-\frac{z^2}{2\sigma^2}\right) \times \sum_{i=0}^{\infty} \frac{L_i^{(-\frac{1}{2})}(z^2/2\sigma^2)(-1/2\sigma^2)^i \mu_M^{(2i)}}{\Gamma[(n/2)+i]}, \quad (26)$$

$$\Pr\{|\mathbf{S}+\mathbf{N}| > a\} = \frac{1}{\Gamma(n/2)} \Gamma\left(\frac{n}{2}, \frac{a^2}{2\sigma^2}\right) - \left(\frac{a^2}{2\sigma^2}\right)^{n/2} \times \exp\left(-\frac{a^2}{2\sigma^2}\right) \sum_{i=1}^{\infty} \frac{L_{i-1}^{(n/2)}(a^2/2\sigma^2)(-1/2\sigma^2)^i \mu_M^{(2i)}}{i\Gamma[(n/2)+i]}, \quad (27)$$

and

$$\Pr\{Z > a\} = \frac{1}{2}\operatorname{erfc}\left(\frac{a}{(2\sigma^2)^{\frac{1}{2}}}\right) - \frac{a}{2}\frac{\Gamma(n/2)}{(2\pi\sigma^2)^{\frac{1}{2}}} \times \exp\left(-\frac{a^2}{2\sigma^2}\right) \sum_{i=1}^{\infty} \frac{L_{i-1}^{(\frac{1}{2})}(a^2/2\sigma^2)(-1/2\sigma^2)^i \mu_M^{(2i)}}{i\Gamma[(n/2)+i]}. \quad (28)$$

The moments $\mu_M^{(2i)} \triangleq E[|\mathbf{S}|^{2i}]$ are determined by the recurrence relations

$$\mu_j^{(2i)} = \sum_{k=0}^{i} \binom{2i}{2k} \frac{c_{n,2k}\, c_{n,2i-2k}}{c_{n,2i}} \mu_{j-1}^{(2k)} \nu_j^{(2i-2k)} \quad (29)$$

for $j = 2, \cdots, M$ with

$$c_{n,2m} = \frac{B[(2m+1)/2,\ (n-1)/2]}{B[\frac{1}{2},\ (n-1)/2]},$$

or by the matrix equation (6).

We next look at some special cases.

A. n = *2-dimensional space*

When $n = 2$, eqs. (24) through (29) reduce in an obvious manner.

The incomplete gamma function in (27) equals (Ref. 10, p. 339):

$$\Gamma\left(1, \frac{a^2}{2\sigma^2}\right) = \exp\left(-\frac{a^2}{2\sigma^2}\right). \tag{30}$$

The Laguerre polynomials $L_i^{(-\frac{1}{2})}$ and $L_i^{(\frac{1}{2})}$ in (26) and (28) can be expressed in terms of Hermite polynomials $H_i(\cdot)$ using the relations (Ref. 10, p. 240):

$$L_i^{(\frac{1}{2})}(x) = \frac{(-1)^i x^{-\frac{1}{2}} H_{2i+1}(x^{\frac{1}{2}})}{i!\,2^{2i+1}} \tag{31a}$$

and

$$L_i^{(-\frac{1}{2})}(x) = \frac{(-1)^i H_{2i}(x^{\frac{1}{2}})}{i!\,2^{2i}}. \tag{31b}$$

For example, eq. (28) can be rewritten as:

$$\Pr\{Z > a\} = \frac{1}{2}\operatorname{erfc}\left(\frac{a}{\sqrt{2\sigma^2}}\right) + \frac{1}{\pi^{\frac{1}{2}}}\exp\left(-\frac{a^2}{2\sigma^2}\right) \times \sum_{i=1}^{\infty} \frac{H_{2i-1}(a/\sqrt{2\sigma^2})(1/2\sigma^2)^i \mu_M^{(2i)}}{(i!)^2 2^{2i}}. \tag{32}$$

We also check that

$$\binom{2i}{2k}\frac{c_{n,2k}\,c_{n,2i-2k}}{c_{n,2i}} = \binom{i}{k}^2$$

so that

$$\mu_2^{(2i)} = \sum_{k=0}^{i}\binom{i}{k}^2 \mu_1^{(2k)}\nu_2^{(2i-2k)}$$

$$= \sum_{k=0}^{i}\binom{i}{k}^2 \nu_1^{(2k)}\nu_2^{(2i-2k)},$$

$$\mu_3^{(2i)} = \sum_{\ell=0}^{i}\sum_{k=0}^{\ell}\binom{i}{\ell}^2\binom{\ell}{k}^2 \nu_1^{(2k)}\nu_2^{(2\ell-2k)}\nu_3^{(2i-2\ell)},$$

and so forth.

Consider the type of problem investigated by Rice[1,3] and Esposito and Wilson[2] in determining the p.d.f.'s of the envelope and instantaneous value of

$$z(t) = \sum_{k=1}^{M} A_k \cos(w_k t + \hat{\theta}_k) + n(t),$$

where each $A_k \geq 0$ is independent of $\hat{\theta}_k$ and $\hat{\theta}_k$ is uniformly distributed on $[0, 2\pi)$. Assume that the pairs $\{(A_k, \hat{\theta}_k)\}$ are independent of each other and of $n(\cdot)$. Suppose $n(t)$ is the result of the passage of zero-mean white stationary Gaussian noise through a bandpass symmetrical

filter. Then $n(t)$ can he written as (Ref. 13, pp. 142–148):

$$n(t) = n_1(t) \cos w_o t - n_2(t) \sin w_o t,$$

where $n_1(t)$ and $n_2(t)$ are zero-mean independent stationary low-pass Gaussian processes with

$$\sigma^2 = E[n(t)]^2 = E[n_1(t)]^2 = E[n_2(t)]^2.$$

Let $\theta_k(t) = (w_k - w_o)t + \hat{\theta}_k$ and thus:

$$\begin{aligned} z(t) &= \sum_{k=1}^{M} A_k \cos [w_o t + \theta_k(t)] + n(t) \\ &= \sum_{k=1}^{M} A_k \cos [w_o t + \theta_k(t)] + n_1(t) \cos w_o t - n_2(t) \sin w_o t \\ &= \left[\sum_{k=1}^{M} A_k \cos \theta_k(t) + n_1(t) \right] \cos w_o t \\ &\qquad\qquad - \left[\sum_{k=1}^{M} A_k \sin \theta_k(t) + n_2(t) \right] \sin w_o t \\ &= A(t) \cos [w_o t + \gamma(t)]. \end{aligned}$$

At any time t_o, let $\theta_k = \theta_k(t_o)$, $n_1 = n_1(t_o)$, and $n_2 = n_2(t_o)$. Put

$$\mathbf{X}_k = (A_k \cos \theta_k, A_k \sin \theta_k), \qquad k = 1, \cdots, M,$$

and

$$\mathbf{N} = (n_1, n_2).$$

Then $\sum_{k=1}^{M} \mathbf{X}_k + \mathbf{N}$ is s.s., so by Theorem 1 it has the form $(\Gamma \cos \Psi, \Gamma \sin \Psi)$, where $\Gamma \geqq 0$ is independent of Ψ and Ψ is uniformly distributed on $[0, 2\pi)$. It follows that

$$\begin{aligned} z(t_o) &= \Gamma \cos \Psi \cos w_o t_o - \Gamma \sin \Psi \sin w_o t_o \\ &= \Gamma \cos (w_o t + \Psi); \end{aligned}$$

that is, $\Gamma = A(t_o)$ and $\Psi = \gamma(t_o)$. Hence, at any time t_o, $A(t_o)$ and $\gamma(t_o)$ are independent and $\gamma(t_o)$ is uniformly distributed on $[0, 2\pi)$. Moreover, the p.d.f. of the "envelope" $A(t_o)$ is the p.d.f. of $\Gamma = |\sum_{k=1}^{M} \mathbf{X}_k + \mathbf{N}|$ which can be determined from (25). Thus we can find the p.d.f. of the envelope of the sum of Gaussian noise plus *any number* of sinusoids with *random* amplitudes and independent uniformly distributed phase angles. The case considered by Esposito and Wilson[2] was that of $M = 2$, $A_1 = a =$ constant and $A_2 = b$ $=$constant, in which case

$$\begin{aligned} \nu_1^{(2m)} &= E[|\mathbf{X}_1|^{2m}] = a^{2m}, \\ \nu_2^{(2m)} &= E[|\mathbf{X}_2|^{2m}] = b^{2m}, \end{aligned}$$

and

$$\mu_2^{(2i)} = \sum_{k=0}^{i} \binom{i}{k}^2 a^{2k} b^{2i-2k}.$$

The envelope p.d.f. is then:

$$p_\Gamma(v) = \frac{v}{\sigma^2} \exp[-v^2/2\sigma^2] \times \sum_{i=0}^{\infty} \frac{L_i(v^2/2\sigma^2)(-1/2\sigma^2)^i}{i!} \left[\sum_{k=0}^{i} \binom{i}{k}^2 a^{2k} b^{2i-2k} \right],$$

which agrees with the result in Esposito and Wilson [Ref. 2, eq. (12)]. This expression was obtained earlier by Goldman.[14]

To find the p.d.f. of $z(t)$ at some time instant t_o, note that:

$$\begin{aligned} z(t_o) &= A(t_o) \cos [w_o t_o + \gamma(t_o)] \\ &= \Gamma \cos (w_o t_o + \Psi). \end{aligned}$$

Since Ψ is uniformly distributed on $[0, 2\pi)$ and the cosine function has period 2π, the p.d.f. of $\Gamma \cos (w_o t_o + \Psi)$ is the same as that of $\Gamma \cos \Psi$. Recall that $(\Gamma \cos \Psi, \Gamma \sin \Psi) = \sum_{k=1}^{M} \mathbf{X}_k + \mathbf{N}$. Thus, $\Gamma \cos \Psi$ is the first component of the 2-dimensional vector $\sum_{k=1}^{M} \mathbf{X}_k + \mathbf{N}$ and, from eq. (26), its p.d.f. is:

$$\begin{aligned} p_{z(t_o)}(z_1) &= p_{\Gamma \cos \Psi}(z_1) \\ &= \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \exp\left(-\frac{1}{2\sigma^2} z_1^2\right) \sum_{i=0}^{\infty} \frac{(-1/2\sigma^2)^i \mu_M^{(2i)}}{i!} \times L_i^{(-\frac{1}{2})}\left(\frac{z_1^2}{2\sigma^2}\right). \end{aligned} \tag{33}$$

In Esposito and Wilson's example, this becomes

$$p_{z(t_o)}(z_1) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \exp\left(-\frac{1}{2\sigma^2} z_1^2\right) \sum_{i=0}^{\infty} \frac{(-1/2\sigma^2)^i}{i!} \times L_i^{(-\frac{1}{2})}\left(\frac{z_1^2}{2\sigma^2}\right) \left[\sum_{k=0}^{i} \binom{i}{k}^2 a^{2k} b^{2i-2k} \right],$$

which agrees with their eq. (29).

We also check that the d.f. in (27) is the same as that obtained in eq. (18) of Ref. 2, when we use the fact that (Ref. 10, p. 241):

$$x L_{i-1}^{(1)}(x) = i[L_{i-1}^{(0)}(x) - L_i^{(0)}(x)].$$

Finally, consider a binary coherent phase-shift-keying communications system operating in the presence of Gaussian noise and M co-channel interferers modeled by a sum of constant amplitude sinu-

soids with independent, uniformly distributed phase angles, $\theta_1, \cdots, \theta_M$. (Details of this model and system may be found in Refs. 4 and 5.) The probability of error in such a system is:[4,5]

$$\text{Pe} = \Pr\left\{\sum_{i=1}^{M} b_i \cos\theta_i + N_1 > a\right\},$$

where $N_1 \sim \mathfrak{N}(0, \sigma^2)$ and a is the amplitude of the transmitted (desired) signal. This probability of error is given by the expression in (32) and agrees with the result found in Refs. 4 and 5. However, eq. (32) can also be used to find the probability of error in this system for a more general class of co-channel interference consisting of a sum of uniformly phased sinusoids having also independent *random* amplitudes.

B. *n = 3-dimensional space*

When $n = 3$, eqs. (24) through (29) reduce in a straightforward way. Equations (24) to (26) and (28) can also be expressed in terms of Hermite polynomials by employing eq. (31). The incomplete gamma function in (27) can be written in terms of tabulated functions by use of the relations (Ref. 10, pp. 339–340):

$$\Gamma(c+1, x) = c\Gamma(c, x) + x^c e^x$$

and

$$\Gamma(\tfrac{1}{2}, x) = \pi^{\frac{1}{2}} \operatorname{erf}(x^{\frac{1}{2}}),$$

where $\operatorname{erf}(\cdot)$ is the error function.

The recurrence relation for the moments becomes, for $M = 2$,

$$\mu_2^{(2i)} = \sum_{k=0}^{i} \binom{2i}{2k} \left(\frac{2k+2}{2k+1}\right)\left(\frac{2i-2k+2}{2i-2k+1}\right)\left(\frac{2i+1}{2i+2}\right) \nu_1^{(2k)} \nu_2^{(2i-2k)},$$

and so on for higher values of M.

IV. SOME COMPUTATIONS

The form of the expressions in (24) through (28) is quite similar, and so the computer programs used for their evaluation were only slight modifications of one basic (Fortran IV) program. Different values of n could also be treated easily. The basic program required computation of a sum of the form:

$$\sum_{i=0}^{\infty} \frac{L_i^{(\alpha)}(x)(-1/2\sigma^2)^i \mu_M^{(2i)}}{\Gamma[(n/2)+i]}, \tag{34}$$

where x is a variable.

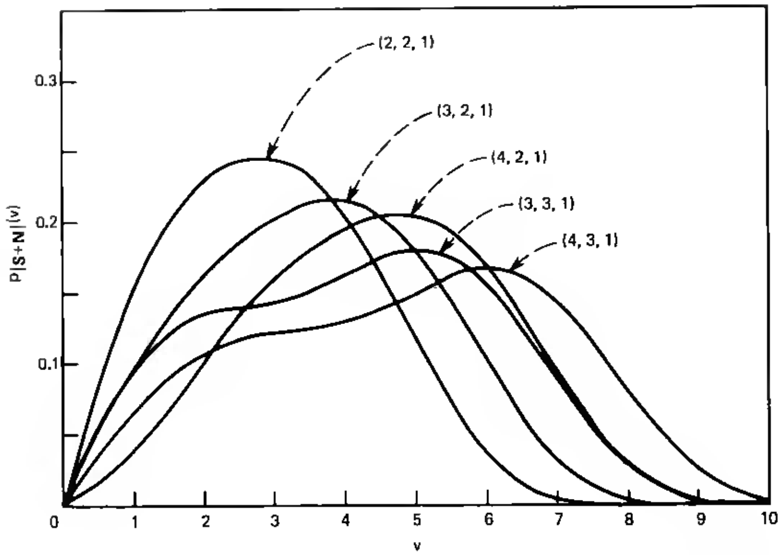

Fig. 1—Plots of p.d.f. of $|\mathbf{S} + \mathbf{N}|$ for $\sigma^2 = 1$, $M = 3$, $n = 2$, and different sets of vector lengths (b_1, b_2, b_3).

In one part of our program, the moments $\mu_M^{(2i)}$ were determined from eq. (29):

$$\mu_j^{(2i)} = \sum_{k=0}^{i} C_n(i, k)\mu_{j-1}^{(2k)}\nu_j^{(2i-2k)}, \qquad j = 2, \cdots, M, \tag{35}$$

where

$$C_n(i, k) \triangleq \binom{2i}{2k}\frac{c_{n,2k}\,c_{n,2i-2k}}{c_{n,2i}}. \tag{36}$$

Using the definition of $c_{n,2m}$ and properties of the beta function, we can show that the coefficients $C_n(i, k)$ are equal to:

$$C_n(i, k) = \frac{\Gamma(i+1)\Gamma(n/2)\Gamma[(n/2)+i]}{\Gamma(k+1)\Gamma(i-k+1)\Gamma[(n/2)+k]\Gamma[(n/2)+i-k]}. \tag{37}$$

To efficiently compute these coefficients and to eliminate "overflow" problems, we utilized the simple recurrence relation

$$C_n(i, k) = \frac{(i-k+1)[(n/2)+i-k]}{k[(n/2)+k-1]}C_n(i, k-1), \qquad k \geqq 1, \tag{38}$$

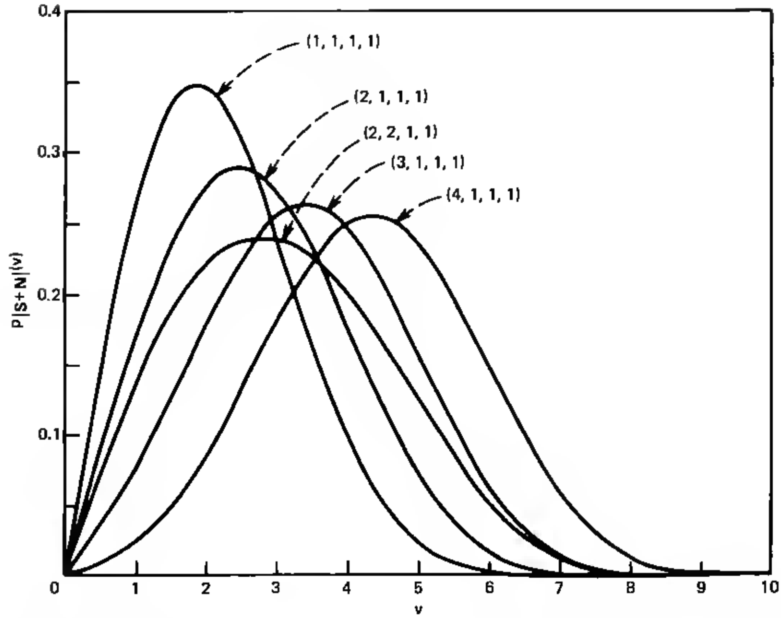

Fig. 2—Plots of p.d.f. of $|S + N|$ for $\sigma^2 = 1$, $M = 4$, $n = 2$, and different sets of vector lengths (b_1, b_2, b_3, b_4).

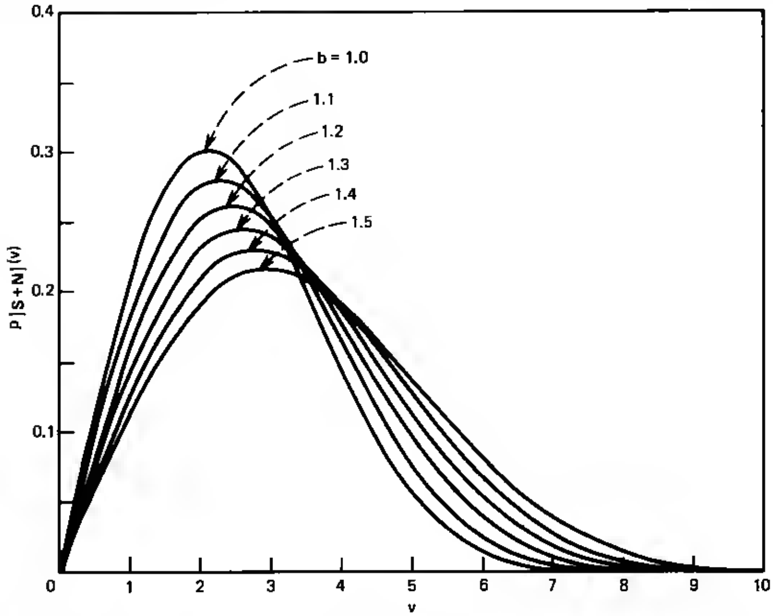

Fig. 3—Plots of p.d.f. of $|S + N|$ for $\sigma^2 = 1$, $M = 6$, $n = 2$, and vector lengths all equal to b.

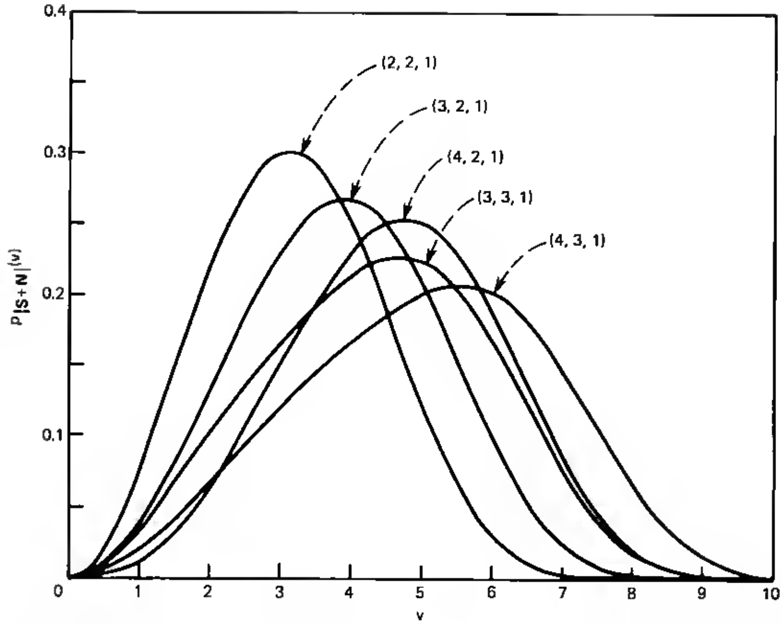

Fig. 4—Plots of p.d.f. of $|\mathrm{S} + \mathbf{N}|$ for $\sigma^2 = 1$, $M = 3$, $n = 3$, and different sets of vector lengths (b_1, b_2, b_3).

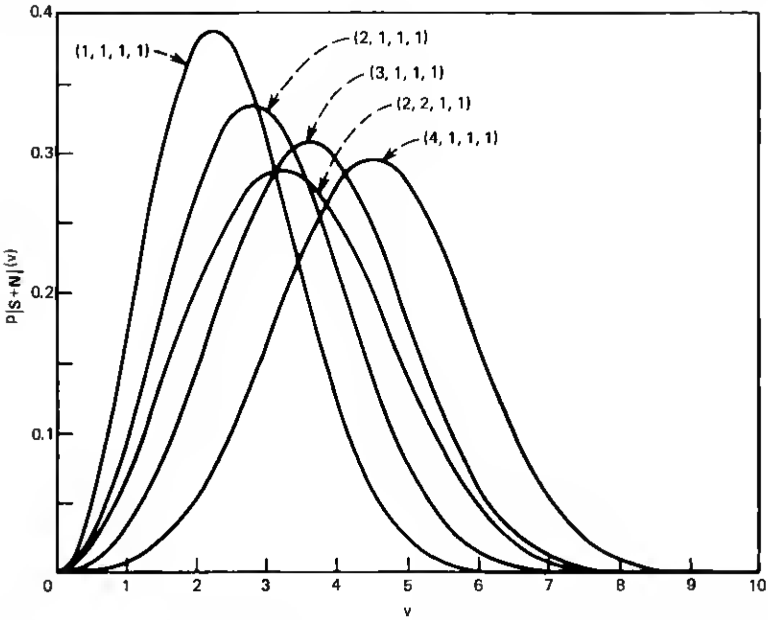

Fig. 5—Plots of p.d.f. of $|\mathrm{S} + \mathbf{N}|$ for $\sigma^2 = 1$, $M = 4$, $n = 3$, and different sets of vector lengths (b_1, b_2, b_3, b_4).

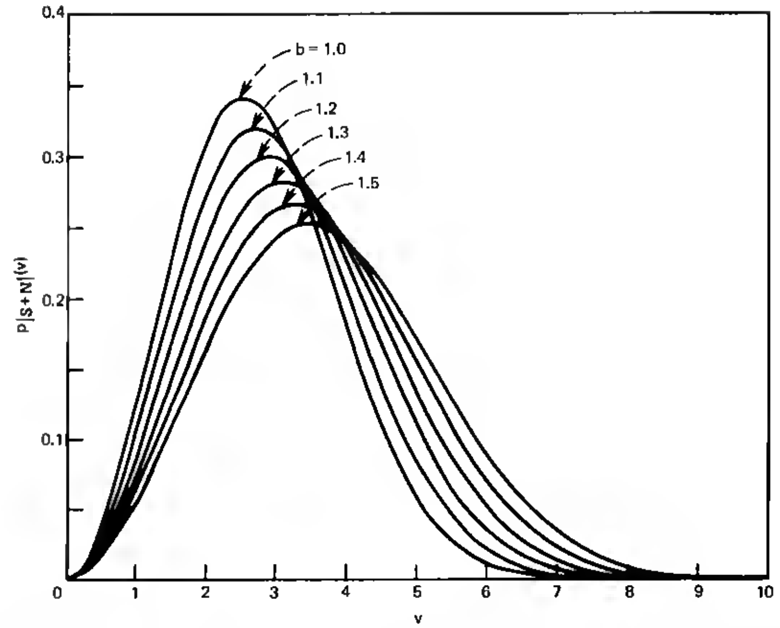

Fig. 6—Plots of p.d.f. of $|\mathrm{S} + \mathrm{N}|$ for $\sigma^2 = 1$, $M = 6$, $n = 3$, and vector lengths all equal to b.

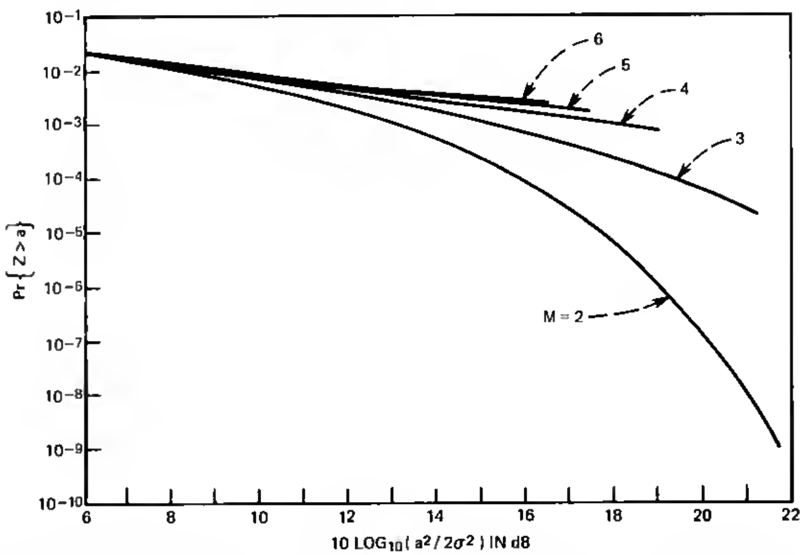

Fig. 7—Plots of Pr $\{Z > a\}$ for $n = 2$, $10 \log_{10} (a^2/\sum_{i=1}^{M} b_i^2) = 6$ dB, $b_1 = \cdots = b_M$, and for various values of M.

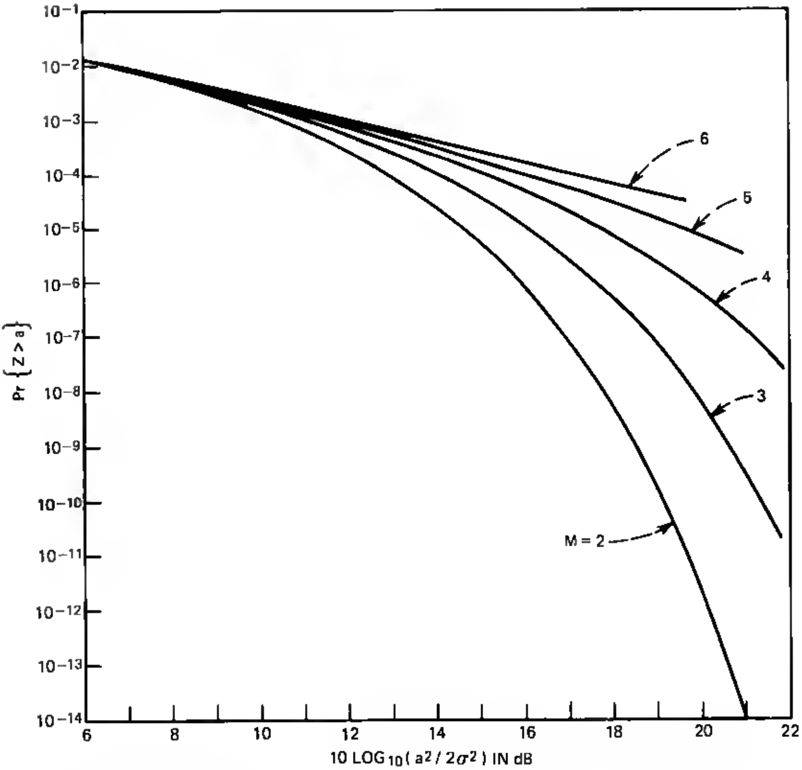

Fig. 8—Plots of Pr $\{Z > a\}$ for $n = 2$, $10 \log_{10} (a^2/\sum_{i=1}^{M} b_i^2) = 8$ dB, $b_1 = \cdots = b_M$, and for various values of M.

together with $C_n(i, 0) \equiv 1$. To evaluate $\mu_M^{(2l)}$ from eq. (35), particular sets of moments $\{\nu_j^{(2l)}\}$ could be read into the program. However, for simplicity we chose spherically symmetric vectors having constant lengths $b_1, \cdots, b_M$.

The second part of the program was concerned with computation of

$$L_i^{(\alpha)}(x)\left(-\frac{1}{2\sigma^2}\right)^i \bigg/ \Gamma\left(\frac{n}{2} + i\right).$$

In order to avoid "overflow" difficulties, we actually computed

$$L_i^{(\alpha)}(x)\lambda^i \bigg/ \Gamma\left(\frac{n}{2} + i\right) \qquad \text{with} \qquad \lambda = -\frac{1}{2\sigma^2}\left(\sum_{i=1}^{M} b_i\right)^2.$$

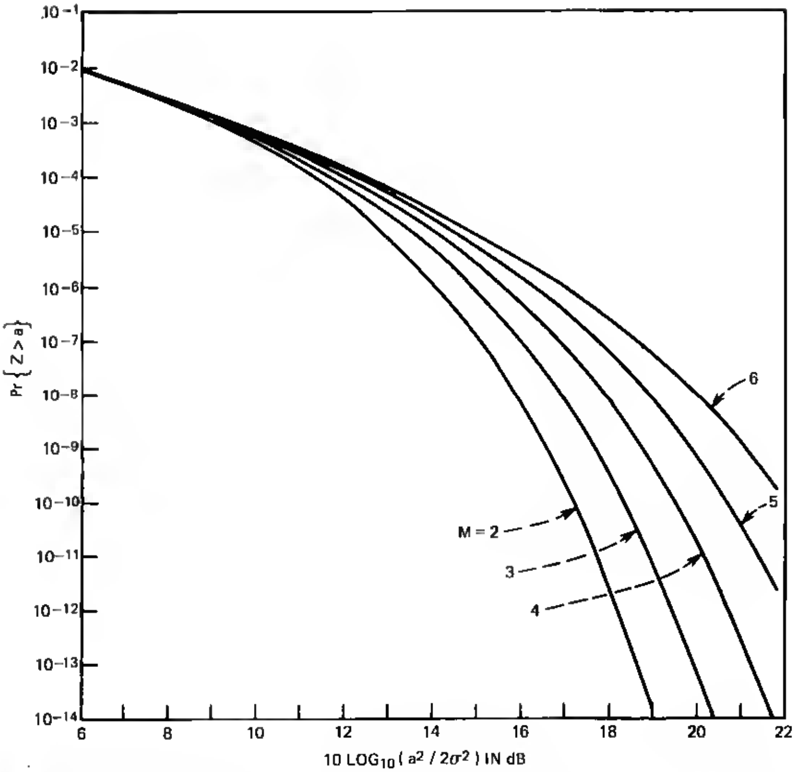

Fig. 9—Plots of Pr $\{Z > a\}$ for $n = 2$, $10 \log_{10} (a^2/\sum_{i=1}^{M} b_i^2) = 10$ dB, $b_1 = \cdots = b_M$, and for various values of M.

To do this efficiently we used the iterative identity:

$$\frac{L_i^{(\alpha)}(x)\lambda^i}{\Gamma[(n/2)+i]} = \frac{(2i+\alpha-1-x)}{i\Gamma[(n/2)+i]}\lambda^i L_{i-1}^{(\alpha)}(x) - \frac{(i+\alpha-1)}{i\Gamma[(n/2)+i]}\lambda^i L_{i-2}^{(\alpha)}(x), \qquad i \geqq 2 \quad (39)$$

[which follows from the Laguerre polynomial recurrence relation (Ref. 10, p. 241)], together with the fact that $L_0^{(\alpha)}(x) = 1$ and $L_1^{(\alpha)}(x) = \alpha + 1 - x$.

The final part of the program computed the sum:

$$\sum_{i=0}^{\infty} \frac{L_i^{(\alpha)}(x)\lambda^i}{\Gamma[(n/2)+i]} \hat{\mu}_M^{(2i)}, \tag{40}$$

where $\hat{\mu}_M^{(2i)} = \mu_M^{(2i)}/(\sum_{i=1}^{M} b_i)^{2i}$. [The factor $1/(\sum_{i=1}^{M} b_i)^{2i}$ was built into

the computation of eq. (35) in order to find $\hat{\mu}_M^{(2i)}$.] A convergence check was provided to end the summation after additional terms did not change any significant digits. Although the program was written to handle up to 200 terms in the sum, many computations required less than 50 terms. As Esposito and Wilson[2] also noted, for certain values of x, σ^2, and $\{b_i\}$, the terms in (40) alternate in sign and have magnitudes of the order 10^{15}. For these cases, precision and convergence could not be guaranteed. The typical CPU time required to compute eq. (40) for 200 values of x was about 10 to 20 seconds in double precision arithmetic on the IBM 370/165 system.

Some representative results of these computations are shown in Figs. 1 to 12. Figures 1 to 6 are plots of $p_{|\mathbf{S}+\mathbf{N}|}(v)$ as a function of v for $\sigma^2 = 1$, for various values of n and M, and for s.s. vectors having constant lengths $b_1, \cdots, b_M$. Curves for $n = M = 2$ were given in Ref. 2. Figures 7 to 12 are plots of $\Pr\{Z > a\}$ versus the quantity $10 \log_{10} (a^2/2\sigma^2)$ for fixed values of the quantity $10 \log_{10} (a^2/\sum_{i=1}^M b_i^2)$ and for various values of n and M. In these curves, for simplicity, we took $b_1 = b_2 = \cdots = b_M$. As we discussed in the last section, the plots in Figs. 7 to 12 represent the probability of error of a binary coherent

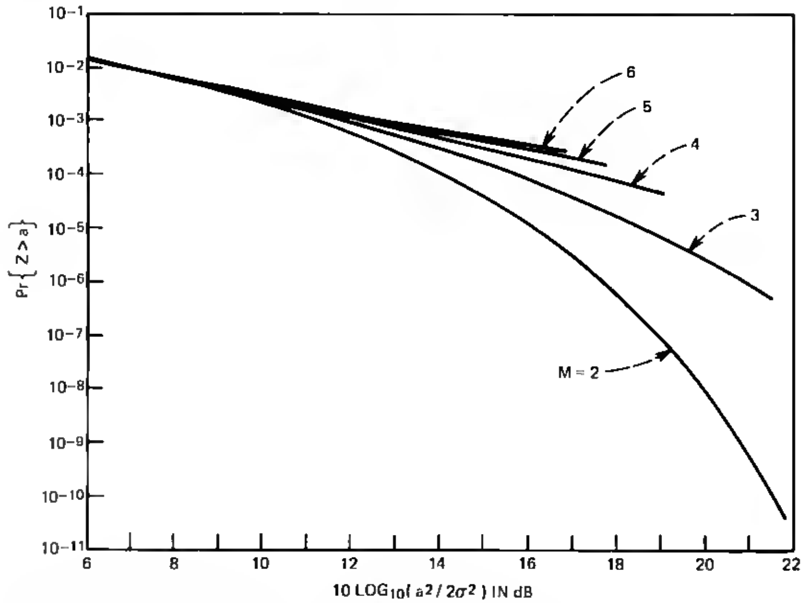

Fig. 10—Plots of $\Pr\{Z > a\}$ for $n = 3$, $10 \log_{10} (a^2/\sum_{i=1}^M b_i^2) = 6$ dB, $b_1 = \cdots = b_M$, and for various values of M.

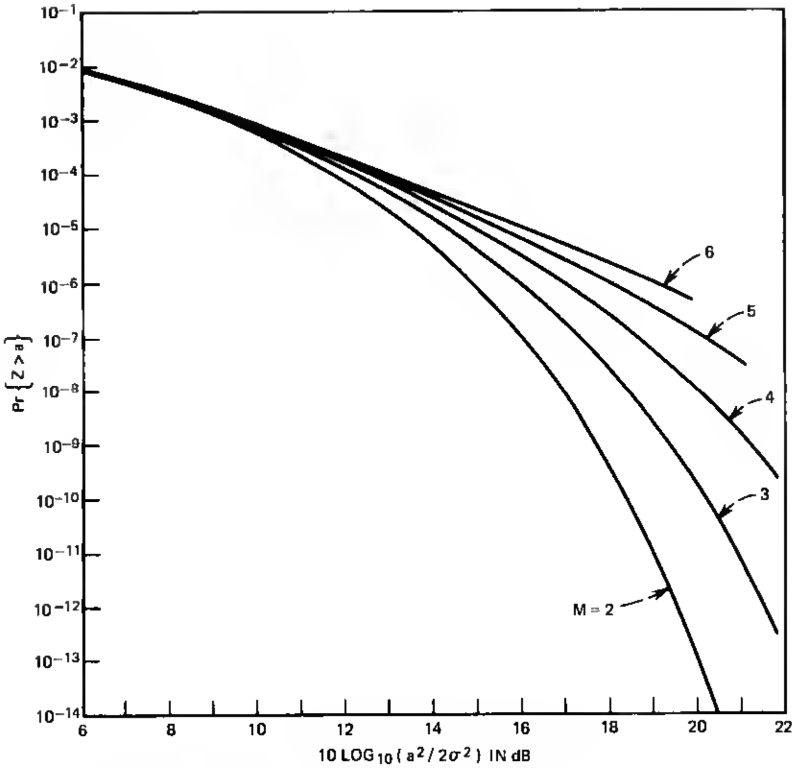

Fig. 11—Plots of Pr $\{Z > a\}$ for $n = 3$, $10 \log_{10} (a^2/\sum_{i=1}^{M} b_i^2) = 8$ dB, $b_1 = \cdots = b_M$, and for various values of M.

phase-shift-keying system versus signal-to-noise ratio,

$$\text{SNR} = 10 \log_{10} (a^2/2\sigma^2) \qquad \text{(dB)},$$

for fixed values of signal-to-interference ratio,

$$\text{SIR} = 10 \log_{10} \left(a^2 \Big/ \sum_{i=1}^{M} b_i^2 \right) \qquad \text{(dB)}.$$

These results extend those previously found in Refs. 4 and 5 to larger values of SNR and smaller values of SIR.

V. CONCLUSION

In this paper we presented expressions for the p.d.f. of a sum of spherically symmetric random vectors plus a Gaussian vector in n-dimensional space. We also found expressions for the p.d.f. and d.f. of the length of this sum and of the projection of this sum onto 1-dimen-

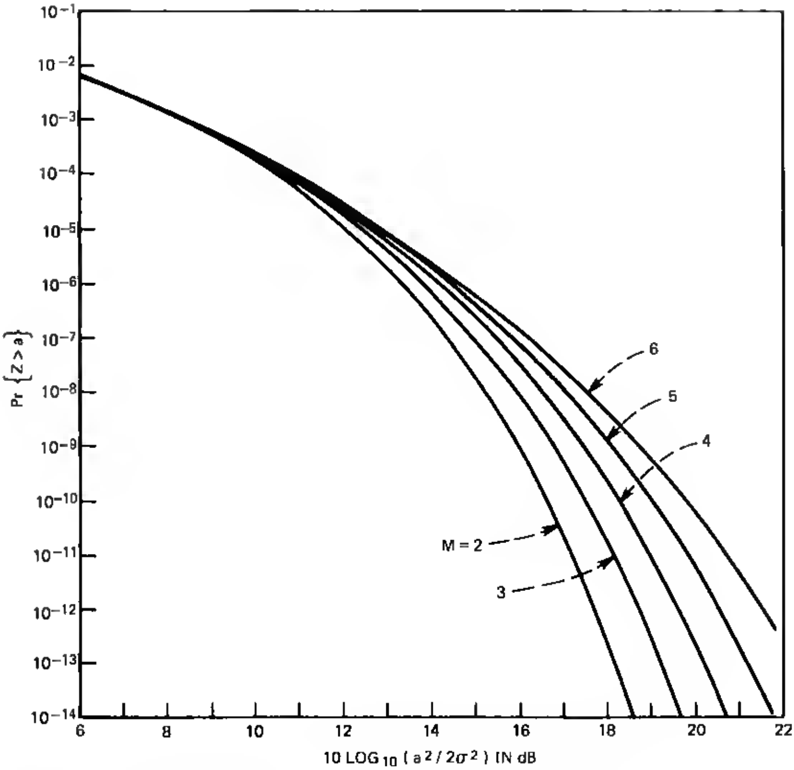

Fig. 12—Plots of Pr $\{Z>a\}$ for $n=3$, $10\log_{10}(a^2/\sum_{i=1}^{M} b_i^2)=10$ dB, $b_1=\cdots=b_M$, and for various values of M.

sional space. All of these expressions were series expansions involving only the moments of the length of the sum of the s.s. vectors. These moments could be found from recurrence relations also derived in the paper. Some computations of the p.d.f.'s and d.f.'s were presented for the 2- and 3-dimensional cases, and an application to a communications system was discussed. However, as pointed out earlier in Refs. 2 and 3, there are sometimes difficulties in evaluating these p.d.f.'s and d.f.'s for certain parameter values, even for the case of s.s. vectors having constant lengths.

APPENDIX

To prove eq. (22) we use the fact (Ref. 10, p. 241) that, for $i \geqq 1$,

$$\frac{d}{dt}\left[e^{-t}t^{\alpha+1}L_{i-1}^{(\alpha+1)}(t)\right] = ie^{-t}t^{\alpha}L_i^{(\alpha)}(t).$$

Integrating this expression over the interval (a, ∞) yields

$$-e^{-a}a^{\alpha+1}L_{i-1}^{(\alpha+1)}(a) = i\int_a^\infty e^{-t}t^\alpha L_i^{(\alpha)}(t)dt. \tag{41}$$

Equation (22) follows from (41) after a simple change of variables.

REFERENCES

1a. S. O. Rice, "Mathematical Analysis of Random Noise (concluded)," B.S.T.J., *24*, No. 1 (January 1945), pp. 46–156.

1b. S. O. Rice, "Statistical Properties of a Sine Wave Plus Random Noise," B.S.T.J., *27*, No. 1 (January 1948), pp. 109–157.

2. R. Esposito and L. R. Wilson, "Statistical Properties of Two Sinewaves in Gaussian Noise," IEEE Trans. Info. Theory, *IT-19*, March 1973, pp. 176–183.

3. S. O. Rice, "Probability Distributions for Noise Plus Several Sine Waves—The Problem of Computation," unpublished work.

4. V. K. Prabhu, "Error Rate Considerations for Coherent Phase-Shift Keyed Systems with Cochannel Interference," B.S.T.J., *48*, No. 3 (March 1969), pp. 743–767.

5. A. S. Rosenbaum, "Binary PSK Error Probabilities with Multiple Cochannel Interferences," IEEE Trans. Comm. Tech., *COM-18*, June 1970, pp. 241–253.

6. K. Yao, "A Representation Theorem and Its Applications to Spherically-Invariant Random Processes," IEEE Trans. Info. Theory, *IT-19*, September 1973, pp. 600–608.

7. I. F. Blake and J. B. Thomas, "On a Class of Processes Arising in Linear Estimation Theory," IEEE Trans. Info. Theory, *IT-14*, January 1968, pp. 12–16.

8. S. Bochner, *Lectures on Fourier Integrals*, Annals of Mathematical Studies, Study 42, Princeton, N.J.: Princeton University Press, 1959.

9. J. Goldman, "Detection in the Presence of Spherically Symmetric Random Vectors," unpublished work.

10. W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, New York: Springer-Verlag, 1966.

11. W. Rudin, *Real and Complex Analysis*, New York: McGraw-Hill, 1966.

12. A. Erdelyi, *et al.*, *Higher Transcendental Functions*, Vol. II, New York: McGraw-Hill, 1953.

13. D. Sakrison, *Communication Theory: Transmission of Waveforms and Digital Information*, New York: John Wiley and Sons, 1968.

14. J. Goldman, "Multiple Error Performance of PSK Systems with Cochannel Interference and Noise," IEEE Trans. Comm. Tech., *COM-19*, August 1971, pp. 420–430.